Solution Set 11

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1 Problem 1

For the harmonic oscillator, the energy *relative to the bottom* (i.e. the ground state) is

$$E_n = \hbar\omega_0 n.$$

a). There is no exclusion principle and therefore all particles will occupy the lowest energy level. Thus

$$E = NE_0 = 0.$$

b). Once again, nothing stops all of the bosons from being in the lowest state, and

$$E = NE_0 = 0.$$

c). The exclusion principle applies and only two electrons can occupy the same energy level. Thus

$$E = 2\hbar\omega_0 \sum_{n=0}^{N/2-1} n = \hbar\omega_0(N)(N/2-1)$$

for and even N and

$$E = 2\hbar\omega_0 \sum_{n=0}^{(N-1)/2} n - (N-1)/2\hbar\omega_0 = 1/2\hbar\omega_0(N)(N/2-1/2)$$

for odd N.

2 Problem 2

a).

$$E = \frac{1}{\sqrt{1-\beta^2}} mc^2$$

Solving for β gives us

$$\beta = \sqrt{1 - (\frac{mc^2}{E})^2}.$$

We can expand this as a Taylor series in $(\frac{mc^2}{E})^2$ to get

$$\beta = 1 - 1/2(\frac{mc^2}{E})^2 + ...$$

where ... denotes higher order terms and can be neglected for $(\frac{mc^2}{E})^2 << 1$.

b,c). Here we have

$$\begin{aligned} \Delta t = \quad & d/v_2 - d/v_1 = d/c(1/(1 - 1/2(\frac{mc^2}{E_2})^2) - 1/(1 - 1/2(\frac{mc^2}{E_1})^2)) = \\ & = d/c\frac{(2E_1^2 - 2E_2^2)m^2c^4}{(E_1^2 - m^2)(E_2^2 - m^2)}. \end{aligned}$$

Using the fact that $E^2 - m^2 \simeq E^2$ we can get

$$m^2c^4 = \frac{2c\Delta t(E_1^2 - E_2^2)}{d(E_1^2 - E_2^2)} = 2.1 * 10^{-10} MeV^2.$$

3 Problem 3

For $s = 0$, $\vec{S_1} \cdot \vec{S_2}/\hbar^2 = -(S_1^2 + S_2^2)/2\hbar^2 = -3/4$ and similarly for $s = 1$, $\vec{S_1} \cdot \vec{S_2}/\hbar^2 = (S^2 - (S_1^2 + S_2^2))/2\hbar^2 = -1/4$. Thus we can use the provided data to get

$$mc^2 = 1578 MeV$$

and

$$\beta^2 = 0.148.$$

Then for $n = 2$

$$E_{s=0} = 3111 MeV$$

and

$$E_{s=1} = 3140.$$

4 Problem 4

An object of mass m will loose the energy of GmM/R when exiting a gravitational well. Thus

$$\frac{\Delta f}{f} = \frac{E' - E}{E} = \frac{-GmM}{ER} = -\frac{GM}{c^2R}.$$

5 Problem 5

a). For radial motion $d\theta = d\phi = 0$ and therefore $v = \frac{dr}{dt}$. Using the fact that light still follows a geodesic (i.e. $ds = 0$) we get

$$\frac{dr}{dt} = gc$$

b). For transverse motion $dr = 0$, and we can choose the coordinates system such that the light is (instantaneously) moving in the θ direction (i.e. $d\phi = 0$). Then

$$v = r\frac{d\theta}{dt} = \sqrt{g}c.$$

Since $g < 1$ the speed of light will be less than c. In a Newtonian picture the light would be accelerated as all massive particles are.

6 Problem 6

a). Here $\gamma_0 \simeq 1$ and therefore

$$\omega \simeq \frac{\omega'}{1+\beta_0}.$$

Solving for β_0 gives us

$$\beta_0 = (\lambda - \lambda')/\lambda' = z.$$

b). Here $1 + \beta_0 \simeq 2$. Therefore

$$\gamma_0 = \frac{\lambda}{2\lambda'} \simeq \frac{\lambda - \lambda'}{2\lambda'} = z/2,$$

where I've used the fact that $\lambda >> \lambda'$.

c). Using the Hubble's law (and the approximation of part a).) we have

$$ct = r = \beta_0 c/H_0 = zc/H_0.$$

Thus $t = 1.4 * 10^5 yr$.

7 Problem 7

Using the fact that $[\hbar] = kg * m^2/s$, we can construct the Planck time

$$\sqrt{G\hbar/c^5},$$

the Planck mass

$$\sqrt{c\hbar/G}$$

and the Planck length

$$\sqrt{G\hbar/c^3}.$$

8 Problem 8

In this problem we can use the facts that $\Omega(t) = \frac{8\pi G\rho(t)}{3H^2(t)}$ and $\rho(t) = \rho(t_0)/r^n$, where $r = R(t)/R(t_0)$, to show that

$$\Omega(t)/\Omega(t_0) = \frac{H^2(t_0)}{H^2(t)r^n}.$$

Also, the equation of the problem 18-12 can be rewritten as

$$\frac{H^2(t)}{H^2(t_0)} - \Omega(t_0)r^{-n} = (1 - \Omega(t_0))r^{-2}.$$

Usin the first equation to eliminate $\frac{H^2(t)}{H^2(t_0)}$ and solving for $\Omega(t)$ will give us the desired result

$$\Omega(t) = \frac{\Omega(t_0)}{\Omega(t_0) + (1 - \Omega(t_0))r^{n-2}}.$$